CONJUGATED HEAT TRANSFER IN THE FLOW OF A NON-NEWTONIAN FLUID WITH VARIABLE PROPERTIES IN A FLAT DUCT

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We solve the problem of the flow of a nonlinearly viscoelastic fluid in the presence of large pressure drops and appreciable nonisothermicity.

Many papers which have appeared on fluid dynamics and heat transfer in the flow of non-Newtonian media in various channels are summarized in detail in [1]. Modeling the flow of such media under conditions of large pressure drops and appreciable nonisothermicity requires treating a system of partial differential equations which cannot be solved exactly. The application of numerical methods hampers the further use of the calculated results, since finding the temperature and flow rate distributions of a fluid in a practical application is only an intermediate step on the way to process optimization. So far no papers have appeared in which an analytic solution for adjoint boundary conditions could take account of the temperature and pressure dependences of the physical properties and the dissipation of energy and nonlinear viscoelasticity simultaneously. An approximate analytic solution of this problem for boundary conditions of the first kind was derived by Tyabin et al. [2], but such heat-transfer conditions are very rarely encountered in real processes.

In the present article we treat the steady flow of a high-viscosity non-Newtonian fluid in a flat duct with a constant temperature T_W on the outer surface. In this case it is necessary to solve the adjoint problem. We investigate media for which slipping at the wall can be neglected. We assume that the temperature of the fluid at the duct inlet is uniform over the cross section and equal to T_0. In accordance with the assumptions made, which are described in [2], the mathematical model of this process can be represented by the following system of equations:

$$\rho v \frac{\partial v}{\partial x} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left[K\left|\frac{\partial v}{\partial y}\right|^{n-1}\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial x}\left[2K\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial y}\right)^{n-1} + \beta_1\left(\frac{\partial v}{\partial y}\right)^2\right], \tag{1}$$

$$\rho c_p v\frac{\partial T_f}{\partial x} = \frac{\partial}{\partial x}\left(\lambda_f\frac{\partial T}{\partial x}\right) + \lambda_f\frac{\partial^2 T_f}{\partial y^2} + \varepsilon T_f v\frac{dp}{dx} + K\left(\frac{\partial v}{\partial y}\right)^{n+1} + 2K\left(\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial v}{\partial y}\right)^{n-1} + \beta_1\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial y}\right)^2, \tag{2}$$

$$v = \frac{\int_0^{h_0} v\,dy}{h_0}, \tag{3}$$

$$\frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2} = 0, \tag{4}$$

$$K(T_f,\ p) = K_{w0}\exp(s\Delta p)\sum_i b_i(\Delta T)^i, \tag{5}$$

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$$\beta_1(T_f,\dot{\gamma}) = \beta_{0w}\left(\frac{\partial v}{\partial y}\right)^{n'-2}\exp(b'\Delta T), \tag{6}$$

$$\lambda(p) = \lambda_0 + \sum_i \xi_i p^i, \tag{7}$$

$$c_p(p) = c_{p0} + \sum_j \gamma_j p^j. \tag{8}$$

The boundary conditions are: $x \geqslant 0,\ y = h_0\quad \frac{\partial v}{\partial y} = 0,\ \frac{\partial T_f}{\partial y} = 0;\ x = 0\ \ T_f = T_0 = \mathrm{const},\ T_s = \varphi(\overline{Y}),$

$$x \geqslant 0,\ y = 0\quad v = 0,\ T_f = T_s,\ \lambda_s\left(\frac{\partial T_s}{\partial \bar{y}}\right)_{\bar{y}=0} = -\lambda_f\left(\frac{\partial T_f}{\partial y}\right)_{y=0}, \tag{9}$$

$$x \geqslant 0,\ \bar{y} = H\quad T_s = T_w = \mathrm{const}.$$

We solve the problem under consideration in the zone of flow by the improved integral method of [3]. The approximate analytic solution employs an auxiliary function $\Delta(X)$ which varies from 0 to 1, has no physical significance, and satisfies the condition

$$\int_0^1 \mathscr{L} d\eta + \int_1^{1/\Delta} \mathscr{L} d\eta = 0, \tag{10}$$

where $\mathscr{L}(T_f) = 0$ is a nonlinear operator, in general form equivalent to the energy equation (2). It was shown in a number of papers [4-6] that the effect of dissipative heating on the temperature profile increases with an increase in the reduced length; i.e., for larger Gratz numbers perturbations of the temperature distribution will occur only near the wall. Therefore, at the first step the second term in Eq. (10) can be neglected, and in the range $0 \le Y \le \Delta$ the temperature profile is specified by a polynomial of arbitrary degree

$$\Theta_f = A_0 + A_1 Y + A_2 Y^2 + \ldots + A_n Y^n, \tag{11}$$

with coefficients which are functions of the longitudinal coordinate and are found from the boundary conditions and supplementary imposed constraints. In this case $\Delta(X)$ is understood as the approximate thickness of the thermal boundary layer. Restricting ourselves in the first approximation to a second-degree polynomial, we have

$$\Theta_f = \Theta_c + 2(1-\Theta_c)\frac{Y}{\Delta} - (1-\Theta_c)\left(\frac{Y}{\Delta}\right)^2. \tag{12}$$

In this equation we have introduced the unknown temperature Θ_c of the inner wall of the duct to satisfy the first matching condition.

By using Eqs. (5), (6), and (12), and averaging the inertial term and the normal stresses over the cross section of the duct in the equation of motion, we obtain expressions for the velocity profile

$$V(Y \leqslant \Delta) = \frac{1}{A}\sum_{i=1}^{9}\frac{B_i}{m+i}\left[(1-Y)^{m+i} - 1\right], \tag{13}$$

$$V(Y \geqslant \Delta) = \frac{1}{A}\left\{\sum_{i=1}^{9}\frac{B_i}{m+i}\left[(1-\Delta)^{m+i} - 1\right] + \left(\frac{K_w}{K_0}\right)^m \frac{1}{m+1}\left[(1-\Delta)^{m+1} - (1-Y)^{m+1}\right]\right\}, \tag{14}$$

$$A = \left(\frac{K_w}{K_0}\right)^m \frac{1-\Delta}{m+2} - \sum_{i=1}^{9}\frac{B_i}{m+i+1}\left[1-(1-\Delta)^{m+i+1}\right],$$

where $B_i = B_i(\Delta)$ [1].

Substituting Eqs. (12)-(14) into the energy equation, and integrating from 0 to Δ, we obtain an ordinary first-order differential equation whose solution enables us to determine $\Delta = \Delta(X)$ under the condition that $\Delta = 0$ at $X = 0$:

$$[F_1(\Delta, X) - F_3(\Delta)] \frac{d\Delta}{dX} = F_4(\Delta, X) + F_2(\Delta, X), \tag{15}$$

where the functions F_1 and F_2 are given in [2], and

$$F_3 = \frac{\mathrm{Br}^*}{\mathrm{Pe}} \frac{1}{A^{n'+1}} \left\{ \sum_{i=1}^{9} \frac{\frac{GB_i}{A} + D_i}{(m+i)(m+mn'+i+1)} \times \right.$$

$$\times [1 - (1-\Delta)^{m+mn'+i+1}] - \sum_{i=1}^{9} \frac{\frac{GB_i}{A} + D_i}{(m+i)(mn'+1)} \times$$

$$\left. \times [1 - (1-\Delta)^{mn'+1}] \right\}; \quad F_4 = \frac{\lambda^*}{\mathrm{Pe}} \frac{4(1-\Theta_c)}{\Delta};$$

$$G = \sum_{i=1}^{9} \frac{D_i}{m+i+1} [1 - (1-\Delta)^{m+i+1}] + \sum_{i=1}^{9} B_i (1-\Delta)^{m+i} + \left(\frac{K_w}{K_0}\right)^m (1-\Delta).$$

The D_i are given arithmetic functions [1].

Equation (15) can be solved by various familiar methods. We performed the integration by the Picard method, using Simpson's rule.

In order to find the unknown temperature $\Theta_c(X)$ it is necessary to examine the temperature distribution on the plane wall of the duct. It is known that the exact solution of Laplace's equation (4) by the Fourier method can be written as an infinite sum of elementary solutions

$$\Theta_s = \sum_{n}^{\infty} \varphi_n(\overline{Y}) f_n(X). \tag{16}$$

Theoretical calculations and experimental data in [7, 8] show that for similar boundary conditions and a high thermal conductivity of the duct wall the temperature distribution in various cross sections is nearly linear; i.e., the temperature profiles are nearly similar. In this case the series (16) converges rapidly; the effect of subsequent terms decreases sharply in comparison with the preceding terms. Therefore, the temperature distribution on the wall can be represented approximately as the product of the first eigenfunctions

$$\Theta_s = \varphi_1(\overline{Y}) f_1(X). \tag{17}$$

The last assumption, independently of the parameter R and the flow conditions, is valid also when the duct wall is thin enough [9].

The function $\varphi_1(\overline{Y})$ is found by solving the Sturm–Liouville problem after substituting (17) into Laplace's equation $\varphi_1'' + \lambda_1\varphi_1 = 0$. The eigenfunction φ_1 and the first eigenvalue λ_1 are found by the Ritz method [10], which determines φ_1 to within a constant.

Thus, the temperature profile is

$$\Theta_s = k\left(1 - \frac{5}{4}\overline{Y} + \frac{1}{4}\overline{Y}^2\right) f_1(X). \tag{18}$$

Using the first matching condition we obtain for $\overline{Y} = 0$

$$\Theta_s = \Theta_c = kf(X).$$

It follows from the second matching condition that

$$\Theta_c = \frac{2}{2 + \frac{5}{4} R\Delta}. \tag{19}$$

After this the temperature and velocity distributions in the flowing fluid in the initial thermal region are completely determined. Using these analytic expressions it is easy to find the local Nusselt numbers or the hydraulic resistance.

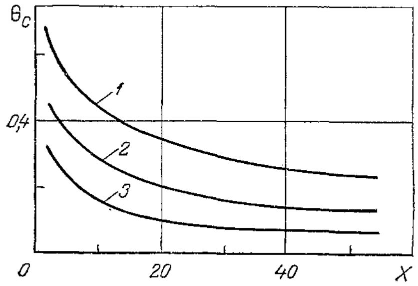

Fig. 1. Temperature variation lengthwise along inner wall of duct: 1) R = 5; 2) 10; 3) 20.

The solution obtained above by the ordinary integral method can be considered only as the first approximation.

If refinement is necessary, according to the Yang scheme the solution of Eq. (15) can be substituted into the initial energy equation, which, after a coordinate transformation [2] goes over into an ordinary second-order differential equation

$$\frac{\lambda^*}{\Delta^2 \mathrm{Pe}} \frac{d^2\Theta_f}{d\eta^2} + Z(\Delta, \eta)\frac{d\Theta_f}{d\eta} = Q(\Delta, \eta), \tag{20}$$

where $Z(\Delta, \eta)$ and $Q(\Delta, \eta)$ are given functions [2]. For the boundary conditions

$$\eta = 0 \quad \Theta_f = \Theta_c, \quad \eta = 1 \quad \Theta_f = 1$$

Eq. (20) can be solved by quadrature

$$\Theta_f = \left(\frac{\partial\Theta_f}{\partial\eta}\right)_{\eta=0} \int_0^{\eta} E^{-1} d\eta + \int_0^{\eta} E^{-1}\left(\int_0^{\eta} QEd\eta\right) d\eta + \Theta_c, \tag{21}$$

$$\left(\frac{\partial\Theta_f}{d\eta}\right)_{\eta=0} = \frac{1 - \int_0^1 E^{-1}\left(\int_0^{\eta} QEd\eta\right) d\eta - \Theta_c}{\int_0^1 E^{-1} d\eta}, \tag{22}$$

$$E = \exp\left(\int_0^{\eta} Zd\eta\right) = \frac{c_p^* \mathrm{Pe}}{\lambda^* A} \frac{\Delta'}{\Delta}\left\{\sum_{i=1}^{9} \frac{B_i}{(m+i)(m+i+1)}[1 - (1-\Delta\eta)^{m+i+1}] - \sum_{i=1}^{9} \frac{B_i}{(m+i)(m+i+2)}[1-(1-\Delta\eta)^{m+i+2}] - \sum_{i=1}^{9} \frac{B_i}{m+i}\frac{\eta^2}{2}\right\} - \frac{c_p^*\Delta\Delta'}{\lambda^*}\frac{\partial\lambda^*}{\partial X}\frac{\eta^2}{2}.$$

The solution in the range $\Delta \le Y \le 1$ for the boundary conditions

$$\eta = 1 \quad \Theta_f = 1, \quad \eta = \frac{1}{\Delta} \quad \frac{1}{\Delta}\frac{d\Theta_f}{d\eta} = 0$$

is sought in a similar way.

If the temperature gradient at the wall is calculated from Eq. (22), a refined value of the local Nusselt number can be determined.

Using the analytic relations obtained, estimates were made of the effects of various factors on the basic laws of heat transfer and resistance for the flow of rheologically complex media in ducts. They were analyzed in detail for boundary conditions of the first kind in [2, 11].

Figure 1 shows the variation of the dimensionless temperature lengthwise along the inner wall of the duct for various values of the parameter R characterizing the ratio of the thermal resistances of the fluid and wall. Such calculations help to estimate the conditions under which the variation of the wall temperature is negligible and can be limited by boundary conditions of the first kind.

Thus, we have obtained an approximate analytic solution of the problem of the flow of a nonlinearly viscoelastic fluid with variable physical properties in a flat duct for adjoining boundary conditions which gives a complete description of the temperature, velocity, and pressure distributions, and can be used in design and technical calculations of equipment used to receive and process non-Newtonian media.

NOTATION

v, T_f, Velocity and temperature of stream; T_s, temperature of duct wall; x, longitudinal coordinate; y, $\bar{y}$, transverse coordinates in stream and on wall, respectively; λ_0 and c_{p0}, thermal conductivity and specific heat at atmospheric pressure; ε, coefficient of thermal expansion; p, hydrostatic pressure; K_{W0}, consistency constant at temperature T_W and atmospheric pressure; s, pressure coefficient of viscosity; n, flow index; b and b', temperature coefficients of viscosity and first difference of normal stresses; b_i, Fourier expansion coefficients of exponential in series of orthogonal Chebyshev polynomials; h_0, half-height of duct; β_1, coefficient of normal stresses; ξ_i and γ_i, empirical constants necessary to approximate experimental data with any degree of accuracy; ρ, density of fluid; H, thickness of duct wall; a, thermal diffusivity; $X = x/h_0$; $Y = y/h_0$; $\bar{Y} = \bar{y}/h_0$; $\Delta = \delta/h_0$; $\eta = Y/\Delta$; $\Theta = (T_f - T_W)/(T_0 - T_W)$; $V = v/\bar{v}$; $m = 1/n$; $c_p^* = c_p/c_{p0}$; $\lambda^* = \lambda/\lambda_0$; $R = \lambda_s/\lambda_f \cdot h_0/H$; $\Delta' = d\Delta/dX$; $Pe = \bar{v}h_0/a_f$; $Br = [K_{W0}\bar{v}^{n+1}]/[\lambda_f(T_0 - T_W)h_0^{n-1}]$; $Br^* = [\beta_{10}\bar{v}^{n+1}]/[\lambda_f(T_0 - T_W)h_0^{n-1}]$; $\Delta T = T_f - T_W$.

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